UDC 539.3

APPROXIMATE REDUCTION OF THE EQUATIONS OF THE THEORY OF ELASTICITY AND ELECTRODYNAMICS FOR INHOMOGENEOUS MEDIA TO THE HELMHOLTZ EQUATIONS

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The vector equations of the dynamic theories of elasticity and electrodynamics, describing the elastic and electromagnetic oscillations in appropriate arbitrarily inhomogeneous isotropic media are considered. The Lamé parameters and density for an elastic medium, as well as the dielectric and magnetic permeability, are assumed continuously differentiable functions of the coordinates. In both cases, the frequency of oscillation is considered a large parameter. It is shown that the elasticity theory equation separates in the zeroth approximation in the oscillation frequency into two uncoupled scalar Helmholtz equations for the longitudinal and the transverse potentials, and the Maxwell equations reduce to a Helmholtz equation for the vecter potential.

1. Oscillations in an arbitrarily inhomogeneous isotropic elastic medium, where the Lamé parameters λ, μ and the medium density ρ are continuously differentiable functions of the coordinates, are described by the dynamic elasticity theory equation

$$Lu = (\lambda + 2\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - \mu\,\mathrm{rot}\,\mathrm{rot}\,\mathbf{u} + \mathrm{grad}\,(\lambda + \mu)\,\mathrm{div}\,\mathbf{u} + \\ + \mathrm{rot}\,(\mathbf{u} \times \mathrm{grad}\,\mu) + \mathrm{grad}\,(\mathrm{grad}\,\mu\mathbf{u}) - \mathbf{u}\Delta\mu + \rho\omega^2\mathbf{u} = 0 \qquad (1.1)$$

For constant λ, μ and ρ Eq. (1.1) decomposes into two scalar Helmholtz equations, one of which describes the longitudinal, and the other the transverse oscillations being propagated with the velocities $v_p = \sqrt{(\lambda+2\mu)/\rho}$ and $v_s = \sqrt{\mu / \rho}$, respectively. Equation (1.1) does not separate in the general case of an arbitrary coordinate dependence of λ, μ and ρ (*).

Let ω be a large parameter relative to all the other quantities of the same dimensionality, for example $\omega \gg |\nabla v_p|$. Then the solution of (1.1) has the form

$$\mathbf{u}(\mathbf{x}, \omega) = e^{i\omega\tau(\mathbf{x})}\mathbf{Q}(\mathbf{x}, \omega) \qquad (1.2)$$

where $e^{i\omega\tau(\mathbf{x})}$ is the most rapidly varying factor in the sense that

$$|\nabla e^{i\omega\tau(\mathbf{x})}| \gg |\nabla \mathbf{Q}(\mathbf{x}, \omega)|$$

For a function of the form (1.2) the operation of differentiation is equivalent, to the accuracy of a factor, to the operation of multiplication by ω. The separation of (1.1) into scalar Helmholtz equations turns out to be possible for such functions to the accuracy of infinitesimal correction terms as $\omega \to \infty$.

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*) For a radially inhomogeneous medium, the separation of (1.1) is achieved successfully by introducing complicated and artificial assumptions about changes in λ, μ and ρ.

The elastic wave field is the sum of the longitudinal $\mathbf{u}_p$ and transverse $\mathbf{u}_s$ fields : $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s$. Let us introduce the longitudinal and transverse potentials φ and ψ as follows (prompted by the corresponding formulas in [1]):

$$\mathbf{u}_p = \operatorname{grad} \varphi + \mathbf{Y}\varphi, \qquad \mathbf{u}_s = \operatorname{rot}(\psi \mathbf{f}) + \mathbf{Z}\psi \tag{1.3}$$

Here $\mathbf{Y}, \mathbf{Z}$ and $\mathbf{f}$ are still arbitrary vectors dependent only on $\mathbf{x}$, and φ and ψ are scalar functions whose form is analogous to (1.2). In conformity with the above, the operation ∇ applied to scalar or vector functions q of the type (1.2), increases the order of their absolute values in ω by one, i.e. $\nabla q = O(\omega q)$. Therefore, the second members in (1.3) are corrections to the first.

Because of the linearity of the operator L, we let it operate separately on the function $\mathbf{u}_p$ represented by the first formula in (1.3), and then extract the Helmholtz operator in the two principal terms and require that the remaining terms, of an order not less than $O(\nabla^2\varphi)$, vanish. Then the expression

$$\mathbf{Y} = \frac{\lambda + 2\mu}{\lambda + \mu} \frac{\nabla\rho}{\rho} - \frac{2\nabla\mu}{\lambda + \mu} \tag{1.4}$$

is obtained for the vector $\mathbf{Y}$ and $L\mathbf{u}_p$ goes over into

$$L(\nabla\varphi + \mathbf{Y}\varphi) = \frac{\lambda + 2\mu}{\sqrt{\rho}} (\nabla + \mathbf{\Phi})(\Delta\varphi^\circ + \omega^2 n_p^2 \varphi^\circ) + o(\nabla^2\varphi) \tag{1.5}$$

$$\varphi^\circ = \varphi\sqrt{\rho}, \quad \mathbf{\Phi} = \frac{\nabla\lambda}{\lambda + 2\mu} - \frac{2\mu\nabla\mu}{(\lambda+\mu)(\lambda+2\mu)} - \frac{\lambda-\mu}{\lambda+\mu}\frac{\nabla\rho}{\rho}, \quad n_p = \frac{1}{v_p}$$

Here n_p is the index of refraction of the longitudinal waves and $o(\nabla^2\varphi)$ is understood to apply to terms less than $|\nabla^2\varphi|$ or, equivalently, than $\omega^2|\varphi|$. Therefore, if the Helmholtz equation for φ° is solved to the accuracy of $o(\nabla\varphi)$ (this is sufficient to obtain the first term of the asymptotic series), then (1.5) is on the order of $o(\nabla^2\varphi)$.

Now, let us substitute the second formula in (1.3) into $L\mathbf{u}_s$, let us extract the Helmholtz operator from the principal terms, and let us require that the remaining terms, of order not less than $O(\nabla^2\psi)$, vanish. We obtain

$$\mathbf{Z} = \mathbf{z}\times\mathbf{f}, \qquad \mathbf{z} = \frac{2\nabla\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}\frac{\nabla\rho}{\rho} \tag{1.6}$$

$$\operatorname{rot}(\nabla\psi\times\mathbf{f}) + \frac{\nabla n_s}{n_s}\times(\mathbf{f}\nabla)\nabla\psi = o(\nabla^2\psi) \tag{1.7}$$

Hence $L\mathbf{u}_s$ is written as follows:

$$L[\operatorname{rot}(\psi\mathbf{f}) + \mathbf{Z}\psi] = \frac{\mu}{\sqrt{\rho}}\{\nabla(\Delta\psi^\circ + \omega^2 n_s^2\psi^\circ)\times\mathbf{f} + $$
$$+ \mathbf{\Psi}(\Delta\psi^\circ + \omega^2 n_s^2\psi^\circ)\} + o(\nabla^2\psi)$$
$$\psi^\circ = \psi\sqrt{\rho}, \quad \mathbf{\Psi} = \operatorname{rot}\mathbf{f} + \frac{\lambda+3\mu}{\lambda+\mu}\frac{\nabla\mu}{\mu} - \frac{3\lambda+5\mu}{2(\lambda+\mu)}\frac{\nabla\rho}{\rho}, \quad n_s = \frac{1}{v_s} \tag{1.8}$$

Here $o(\nabla^2\psi)$ has the same meaning as in (1.5). If ψ° is the solution of the Helmholtz equation in zeroth approximation, then (1.8) is on the order of $o(\nabla^2\psi)$. Substituting (1.5) and (1.8) into (1.1), we obtain

$$\frac{\lambda+2\mu}{\sqrt{\rho}}(\nabla + \mathbf{\Phi})(\Delta\varphi^\circ + \omega^2 n_p^2\varphi^\circ) + \frac{\mu}{\sqrt{\rho}}\{\nabla(\Delta\psi^\circ + \omega^2 n_s^2\psi^\circ)\times\mathbf{f} + $$
$$+ \mathbf{\Psi}(\Delta\psi^\circ + \omega^2 n_s^2\psi^\circ)\} + o(\nabla^2\varphi) + o(\nabla^2\psi) = 0 \tag{1.9}$$

If φ° and ψ° are the solutions of the appropriate Helmholtz equations to $o(\nabla\varphi)$ and $o(\nabla\psi)$ accuracy, then (1.9) is satisfied to the accuracy of $o(\nabla^2\varphi)+o(\nabla^2\psi)$. Hence, (1.1) is satisfied to the accuracy $o(\Delta\mathbf{u})$ and, therefore, $\mathbf{u}=\nabla\varphi+\operatorname{rot}(\psi\mathbf{f})$ yields the zeroth approximation of the solution of the dynamic elasticity theory equation.

Therefore, the vector equation (1.1) reduces to two Helmholtz equations

$$\Delta\varphi^\circ+\omega^2 n_p{}^2\varphi^\circ=0, \qquad \Delta\psi^\circ+\omega^2 n_s{}^2\psi^\circ=0 \tag{1.10}$$

whose solutions, substituted into (1.3), yield the solution of (1.1) in zeroth approximation. It should be noted that the second members in (1.3) are corrections and yield no contribution in zeroth approximation of the field $\mathbf{u}$, although they play an essential part in going from (1.1) to (1.9).

If the ray solution of (1.1) is of interest, i.e. $Q(\mathbf{x},\ \omega)$ which is a series in reciprocal powers of the parameter ω, then the field $\mathbf{u}$ is defined by the formula

$$\mathbf{u}=[\nabla\varphi+\operatorname{rot}(\psi\mathbf{f})]\left[1+O\left(\frac{1}{\omega}\right)\right] \tag{1.11}$$

where φ and ψ are the ray solutions of (1.10). Also valid for the field in the shadow is (1.11), in which $O(1/\ \omega)$ is replaced by $O(1/\omega^{1/3})$, and φ and ψ are the solutions of (1.10) in the shadow domain.

Let us explain what the vector $\mathbf{f}$ should be so that (1.7) would be satisfied. Let ψ be

$$\psi(\mathbf{x},\omega)=e^{i\omega\tau_s(\mathbf{x})}\,\psi_1(\mathbf{x},\omega)$$

where $\psi_1(\mathbf{x},\ \omega)$ is a slowly varying function, and $\tau_s(\mathbf{x})$ is the eikonal of the transverse wave, i.e. the equation $(\nabla\tau_s)^2=n_s{}^2$ is valid. Then

$$\nabla\psi=i\,\omega\nabla\tau_s\psi+o(\nabla\psi)$$

and (1.7) goes over into the following:

$$\nabla\tau_s\times(\nabla\tau_s\nabla)\,\mathbf{f}+(\mathbf{f}\nabla\tau_s)\frac{\nabla n_s}{n_s}\times\nabla\tau_s=0 \tag{1.12}$$

The vector $\mathbf{f}$ written in a local coordinate system connected to the ray

$$\mathbf{f}=a_f\nabla\tau_s+\sin\theta\mathbf{n}-\cos\theta\mathbf{b} \tag{1.13}$$

satisfies (1.12), where a_f is an arbitrary coefficient, θ is an angle characterizing twisting of the ray, $\mathbf{n}$ is the normal and $\mathbf{b}$ the binormal to the ray at a given point. Substituting (1.13) into the second relation in (1.3), we obtain

$$\mathbf{u}_s=i\omega n_s\psi(\cos\theta\mathbf{n}+\sin\theta\mathbf{b})+o(\omega\psi) \tag{1.14}$$

i.e. the transverse wave field in zeroth approximation should be polarized in a plane perpendicular to the propagation direction, which corresponds to the ray formula [2].

An illustration of the application of this approach to the solution of the problem of diffraction by the smooth interface of two arbitrary inhomogeneous elastic media is presented in [3].

2. Let us consider the Maxwell equations for an inhomogeneous and isotropic medium

$$\operatorname{rot}\mathbf{H}=-\frac{i\omega}{c}\varepsilon\mathbf{E}, \qquad \operatorname{rot}\mathbf{E}=\frac{i\omega}{c}\mu\mathbf{H} \tag{2.1}$$

$$\operatorname{div}\mu\mathbf{H}=0, \qquad \operatorname{div}\varepsilon\mathbf{E}=0 \tag{2.2}$$

where $\varepsilon=\varepsilon(\mathbf{x})$ and $\mu=\mu(\mathbf{x})$ are the dielectric and magnetic permeabilities.

One equation is easily obtained from (2.1) for the magnetic intensity vector, say:

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = \frac{\omega^2}{c^2}\varepsilon\mu\mathbf{H} + \frac{1}{\varepsilon}\nabla\varepsilon\times\operatorname{rot}\mathbf{H} \tag{2.3}$$

For constant ε and μ Eq. (2.3) goes over into the Helmholtz equation, which does not hold if ε and μ are arbitrary functions of the coordinates. However, even in this latter case if ω is considered a large parameter and the solution is sought in the form (1.2), Eq. (2.3) can be reduced, in a zeroth approximation, to the Helmholtz equation.

Let

$$\mathbf{H} = \operatorname{rot}\frac{\mathbf{A}}{\sqrt{\mu}} + \frac{1}{\sqrt{\mu}}\mathbf{C}\times\mathbf{A} \tag{2.4}$$

where the vector potential $\mathbf{A}$ is a function of the form (1.2), and $\mathbf{C}$ is still an arbitrary vector. Let us substitute (2.4) into (2.3), let us extract the Helmholtz operator applied to the vector $\mathbf{A}$ from the first two terms and let us require that the remaining terms, not less than $O(\nabla^2\mathbf{A})$ in order, vanish. We then obtain $\mathbf{C} = \nabla\mu / \mu$, and the equation for the potential is

$$\frac{\nabla n}{n}\times\operatorname{grad}\operatorname{div}\mathbf{A} = o(\nabla^2\mathbf{A}) \tag{2.5}$$

Hence, (2.3) goes over into the following:

$$\operatorname{rot}(\Delta\mathbf{A} + \omega^2 n^2\mathbf{A}) - \left(\frac{\nabla\varepsilon}{\varepsilon} + \frac{1}{2}\frac{\nabla\mu}{\mu}\right)\times(\Delta\mathbf{A} + \omega^2 n^2\mathbf{A}) + o(\nabla^2\mathbf{A}) = 0 \tag{2.6}$$

where $n = \sqrt{\varepsilon\mu} / c$ is the index of refraction. If the potential $\mathbf{A}$ satisfies the Helmholtz equation to $o(\nabla\mathbf{A})$ accuracy (which assures obtaining the first term of the asymptotic series), then (2.6) is satisfied to $o(\nabla^2\mathrm{A})$ accuracy. It hence follows that $\mathbf{H} = \mu^{-1/2}\operatorname{rot}\mathbf{A}$ satisfies (2.3) to $o(\nabla\mathbf{H})$ accuracy, and yields the zeroth approximation of the solution.

As in Sect. 1, the second member in (2.4) is a correction and yields no contribution to the zeroth approximation. Equation (2.5) is satisfied if the vector potential has the form (1.2), where $\tau(x)$ is the eikonal of the electromagnetic wave, and the vector $\mathbf{Q}$ in zeroth approximation is polarized in a plane perpendicular to the direction of propagation.

Let us note that if $\mathbf{H}$ in the form (2.4) is substituted into the first of Eqs. (2.2), where $\mathbf{C} = \nabla\mu / \mu$, it is satisfied identically.

Equation (2.3) can also be reduced to a Helmholtz equation for the scalar potential if it is considered that $\mathbf{A} = \psi\mathbf{f}$ in (2.4). In this case the operator (2.3) goes over into (1.8), where $\psi^\circ = \sqrt{\mu}\psi$, $\Psi = \operatorname{rot} f - \nabla\varepsilon / \varepsilon - {}^1/_2\nabla\mu/\mu$ and the index of refraction of the electromagnetic wave n takes the place of n_s in the Helmholtz operator. Equation (1.7), solved exactly as in the case of a transverse elastic wave, is obtained for the vector $\mathbf{f}$. On the other hand, a vector potential can be introduced instead of the scalar potential for the field $\mathbf{u}_s$ by using formulas analogous to (2.4), and the subsequent analysis is exactly the same as for electromagnetic waves. Consequently, we arrive at the Helmholtz equation for the transverse vector potential.

In conclusion, let us note that other linear equations of mathematical physics, such as the magnetohydrodynamics and magnetoelasticity equations, can apparently also be reduced to Helmholtz equations in zeroth approximation.

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ON THE REDUCTION OF INTEGRAL EQUATIONS OF THE THEORY OF ELASTICITY TO INFINITE SYSTEMS

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We indicate formal methods for the reduction of the integral equations of the theory of elasticity (not considered in [1]) to infinite systems of algebraic equations. We consider an integral equation of the first kind with a difference kernel of mixed type, i.e. containing both Fredholm and Volterra operators. To such an equation one can reduce, for example, the problem of the bending of a semi-infinite plate on a linearly deformable foundation when for the inversion of the differential operator one makes use of the Cauchy function rather than the Green function [2]. The method by which this equation is reduced to an infinite system is based on the presence of spectral relations for the semi-infinite interval. In addition to the relations of similar type, indicated in [3], new spectral relations on the semi-infinite interval are constructed. An integral equation of the second kind and of mixed type is considered. Integral equations of the first and second kind with difference kernels and data prescribed on the axis with a cut-off segment are studied. We consider an integral equation of the second kind on a finite interval with a kernel represented through an improper integral of the product of Bessel functions. The suggested methods can be carried over to the corresponding systems of integral equations.

1. Let us consider the integral equation of mixed type

$$\int_0^\infty k(x-y)\varphi(y)\,dy + \int_0^x l(x-y)\varphi(y)\,dy = f(x) \tag{1.1}$$

We will assume that the integral representation

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(t)\,e^{-ixt}\,dt \tag{1.2}$$